squared (none of our "units" change in value when squared), and but little ingenuity is required to combine a set of four of them in such a way as to fulfill all the stated requirements. One such set is

$$\left.\begin{aligned} H&=\tfrac{1}{2}[1-j-\theta(i+k)]\\ I&=\tfrac{1}{2}[1+j+\theta(i-k)]\\ J&=\tfrac{1}{2}[1+j-\theta(i-k)]\\ K&=\tfrac{1}{2}[1-j+\theta(i+k)]\ \text{whence}\\ 1&=\tfrac{1}{2}(H+I+J+K)\\ i&=\frac{\theta}{2}(H-I+J-K)\\ j&=\tfrac{1}{2}(-H+I+J-K)\\ k&=\frac{\theta}{2}(H+I-J-K) \end{aligned}\right\} \qquad (A)$$

where $\theta=\sqrt{-1}$, and i, j, ank k are Hamilton's unit vectors. Other similar combinations can readily be obtained from this set by a cyclical permutation of the unit vectors, i, j, k, or by substituting for these vectors other rectangular unit vectors. All such combinations must, however, be considered as mere variations of (A). We know that in ordinary algebra we may substitute such operators as d/dx, d/dy, d/dz, etc., for ordinary algebraic numbers because their laws of combination are identical. Whether, likewise, such distinct values of H, I, J, and K exist is a matter of speculation, but seems not impossible.

It might have been a more logical and simple process to have commenced by assuming (A) and then deducing the "multiplication table" therefrom. Such a method of procedure would have led up to the same results, but would not have made it so evident that the values of the fundamental "units" given in (A) are mere *particular* values.

By closely examining (A) we learn that this interpretation of our "units" implies the following:

First. Each "unit" is a Cayleyan nullitat (quaternion with zero norm).

Second. All quaternions, including scalars and vectors, can be written in terms of H, I, J, and K, and vice versa.

Third. These "units" may be algebraically combined in accordance with the same laws that govern the combination of quaternions. (If division and the law of indices be excepted, there seems to be no reason why the laws of quaternions should not govern in the combination of other entirely distinct interpretations of these "units," should such interpretations be found.)

Let $Q=(aH+bI+cJ+dK)$, $R=(eH+fI+gJ+hK)$, $W=(rH+sI+tJ+uK)$, a, b, c, d, e, f, g, h, r, s, t, and u, being numbers in the algebraic

field, operators, such as d/dx, d/dy, d/dz, etc., or a combination of such numbers and operators. Then, if $Q.R=W$,

$$r=e(a+c)+f(a-c),\ s=e(b-d)+f(b+d),$$
$$t=g(c+a)+h(c-a),\ \text{and}\ u=g(d-b)+h(d+b).$$

Using ordinary quaternion methods and Hamilton's notation,

$$SQ=\tfrac{1}{2}(a+b+c+d),$$
$$VQ=(Q-SQ)=(a-w)H+(b-w)I+(c-w)J+(d-w)K,$$
$$[w\equiv\tfrac{1}{4}(a+b+c+d)],$$
$$TQ=\sqrt{[2(ad+bc)]},\ \text{and}$$
$$UQ=(2ad+2bc)^{-\frac{1}{2}}(aH+bI+cJ+dK).$$

Similar expressions can readily be found for SUQ, VUQ, TVQ, $TVUQ$, etc. It would, indeed, be possible to construct an entire system of quaternions in which H, I, J, K would be the "units" used to replace unity and the i, j, k of Hamilton. Such a system would necessitate the use of irrational numbers in expressions involving ordinary vectors, and would, therefore, not be well suited for use in problems which concern the physicist; it would, however, have the remarkable characteristic that scalars, vectors, and vector products could all alike be expressed in terms of the same linear, homogeneous units, although such expressions would be cumbersome and of little practical value to the physicist. It is a matter of historic interest that Hamilton was much concerned about the heterogeneous character of his vector products, and tried to find an "*extra-spacial*" *unit* which would render such products homogeneous.* The $\sqrt{-1}$ of algebra may, perhaps, not be considered such an "*extra-spacial*" *unit*, but may evidently be used to obtain the result Hamilton sought to obtain by the use of such a unit. Inasmuch as the heterogeneity of vector multiplication disappears when $\sqrt{-1}$ is introduced by the linear substitutions involved in these "units," we may, perhaps, better conclude that the heterogeneity which Hamilton sought to remove was, after all, only a seeming heterogeneity resulting from the point of view afforded by his particular system of fundamental unit vectors, whereas quaternion analysis as a system is, of course, independent of any particular set of fundamental unit vectors.

But, in conclusion, we may observe that the algebraic properties of these "units" had been defined and proven consistent before any attempt was made to give them an interpretation. Had we, indeed, entirely failed to find an interpretation, these "units" would, nevertheless, have been realities, in an abstract sense, to the pure algebraist. The fact that they happen to be isomorphic to a certain set of four imaginary quaternions is, of course, not without interest to him, but should be looked upon by him as a mere coincidence.

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*See article on Quaternions by Professor Tait in *Encyclopedia Brittannica.*

CONDITION IN TERMS OF THE INVARIANTS OF THE QUARTIC THAT ITS FOUR DISTINCT ROOT-POINTS BE CONCYCLIC.

By DR. T. E. McKINNEY, Wesleyan University, Middletown, Conn.

The necessary and sufficient condition that the root-points of the quartic

$$(1) \qquad \sum_{i=0}^{i=4} a_i z^{4-i}, \ a_i \text{ complex},$$

be concyclic is that an anharmonic ratio of the roots of the quartic be real. The equation giving the six anharmonic ratios of these four roots is

$$(2) \quad t^6 - 3t^5 + (6 - \frac{I^3}{\Delta})t^4 - (7 - \frac{2I^3}{\Delta})t^3 + (6 - \frac{I^3}{\Delta})t^2 - 3t + 1 = 0,$$

where $a_0^2 I = a_2^2 - 3a_1 a_2 + 12 a_0 a_4$;

$$a_0^3 I_1 = 27(a_1^2 a_4 + a_0 a_3^2) - 9 a_1 a_2 a_3 - 72 a_0 a_2 a_4 + 2 a_2^3,$$

$$27 a_0^6 \Delta = 64 I^3 - I_1^2.$$

The discriminant D of equation (2) is

$$D = \Pi (t_i - t_j)^2, \ i = 1, 2, \ldots, 5; \ j = i+1, \ldots, 6,$$

$$= \frac{I^{12}}{\Delta^4} (4 \frac{I^3}{\Delta} - 27)^3.$$

Every root of equation (2) is a rational function with real coefficients of every other, so that the roots are either all real or all complex. When the roots are all real I^3/Δ is real. When the roots are not only real but also distinct, $D > 0$. When I^3/Δ is real, and $D > 0$, equation (2) has an even number of pairs of conjugate roots. Hence two and, therefore, all roots are real. This result may be expressed as follows:

THEOREM. *The necessary and sufficient condition that the four distinct root-points of the quartic*

$$\sum_{i=0}^{i=4} a_i z^{4-i}, \ a_i \text{ complex},$$

be concyclic is that $\frac{4I^3}{\Delta} - 27 > 0$, *where* I *and* Δ *are the invariants of the quartic.*

SUMMATION OF CERTAIN INFINITE SERIES.

By W. J. GREENSTREET, M. A., F. R. A. S., Editor of The Mathematical Gazette, Stroud, England.

I note with some surprise that no solution of Problem 221 has yet (November, 1906) reached the MONTHLY. I therefore take the liberty of pointing out the method usually adopted in questions of this type, and embody in my remarks solutions of many similar questions taken from Todhunter and Hogg's Trigonometry, Hobson's Trigonometry (our best English work on the subject), and from various Cambridge Scholarship and other papers of recent years. The whole will, I hope, form a useful summary for those to whom the methods are unfamiliar.

1. To save space, we write the series $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$ to ∞ in the form $\sum_{n=1}^{n=\infty}\frac{1}{n^2}$; $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots$ to ∞, in the form $\sum_{n=1}^{n=\infty}\frac{1}{(2n-1)^2}$.

The sign $\equiv$ is used for "identical with." We know that

$$\left.\begin{aligned}\frac{\sin\theta}{\theta}&=1-\frac{\theta^2}{3!}+\frac{\theta^4}{5!}-\frac{\theta^6}{7!}+\ldots\equiv\sum_{n=1}^{n=\infty}(-1)^{n-1}\frac{\theta^{2n-2}}{(2n-1)!}\\ \text{and also }&=\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\ldots\equiv\prod_{n=1}^{n=\infty}\left(1-\frac{\theta^2}{n^2\pi^2}\right)\end{aligned}\right\}(A).$$

Taking logarithms of the two right hand expressions, and expanding in powers of θ, we may equate the coefficients of the respective powers of θ in the system (A).

Thus equating coefficients of θ^2 we have

$$(a)\qquad -\frac{\theta^2}{3!}=-\theta^2\left(\sum_{n=1}^{n=\infty}\frac{1}{n^2\pi^2}\right),\ \text{whence}\ \sum_{n=1}^{n=\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$$

Hence, if $a, b, c, \ldots$ denote all the prime numbers except unity,

$$\begin{aligned}(b)\qquad &(1-a^{-2})^{-1}(1-b^{-2})^{-1}(1-c^{-2})^{-1}\ldots\\ &=(1+a^{-2}+a^{-4}+\ldots)(1+b^{-2}+b^{-4}+\ldots)(1+c^{-2}+c^{-4}+\ldots)\ldots\\ &=1+(a^{-2}+b^{-2}+c^{-2}+\ldots)+\ldots+(a^{-r}b^{-s}\ldots)^2+\ldots\\ &=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}\ldots\ ad\ infinitum=\frac{\pi^2}{6}.\end{aligned}$$

We also have at once $\prod_{2}^{\infty}(1-a^{-2}) = \frac{6}{\pi^2}$.

Again equating coefficients of θ^4 in (A), we have

$$\frac{1}{5!} - \tfrac{1}{2}\left(\frac{1}{3!}\right)^2 = -\frac{1}{2\pi^4}\sum_{n=1}^{n=\infty}\frac{1}{n^4},$$

and it follows that

(c) $$\sum_{n=1}^{n=\infty}\frac{1}{n^4} = \frac{\pi^4}{12}\left(\tfrac{1}{3}-\tfrac{1}{5}\right) = \frac{\pi^4}{90}.$$

Then, if $a, b, c, \ldots$ are the prime numbers we have

(d) $$(1-a^{-4})^{-1}(1-b^{-4})^{-1}(1-c^{-4})^{-1}\ldots$$

$$= (1+a^{-4}+a^{-8}+\ldots)(1+b^{-4}+b^{-8}+\ldots)(1+c^{-4}+c^{-8}+\ldots)$$

$$= (1+a^{-4}+b^{-4}+\ldots+(a^r b^s c^t\ldots)^4+\ldots$$

$$= \sum_{n=1}^{n=\infty}\frac{1}{n^4} = \frac{\pi^4}{90} \text{ by } (c).$$

Thus dividing (d) by (b) we have

(e) $\prod_{2}^{\infty}(1+a^{-2}) = \frac{15}{\pi^2}$, which is in No. 221, p. 190. And

$$\prod_{2}^{\infty}(1+a^{-2})^{-1} \equiv \frac{2^2}{2^2+1}\cdot\frac{3^2}{3^2+1}\cdot\ldots = \frac{\pi^2}{15}.$$

We can connect up the series $\sum_{n=1}^{n=\infty}\frac{1}{n^2}$ with $\sum_{n=1}^{n=\infty}\frac{1}{(2n-1)^2}$, as follows:

Let $S = \frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\ldots$

$$S_1 = \frac{1}{1^n}+\frac{1}{3^n}+\frac{1}{5^n}+\frac{1}{7^n}+\ldots$$

Then $S = \frac{1}{1^n}+\frac{1}{3^n}+\frac{1}{5^n}+\ldots+\frac{1}{2^n}\left(\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\ldots\right) = S_1 + \frac{1}{2^n}S$.

(f) $\therefore S_1 = \frac{2^n-1}{2^n}S$. So that if S be known, S_1 is known. For instance, if $n=2$, $S = \frac{1}{6}\pi^2$ by (a),

(*g*) and $S_1 \equiv \sum_{n=1}^{n=\infty} \frac{1}{(2n-1)^2} = \frac{2^2-1}{2^2} S = \frac{3}{4} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{8}$.

If $n=4$, $S = \frac{\pi^2}{90}$ by (*c*),

(*h*) and $S_1 \equiv \sum_{n=1}^{n=\infty} \frac{1}{(2n-1)^4} = \frac{2^4-1}{2^4} S = \frac{15}{16} \cdot \frac{\pi^4}{90}$ by (*c*) $= \frac{\pi^4}{96}$.

(*i*) Consider $\sum_{n=1}^{n=\infty} (-1)^{n-1} \sum_{n=1}^{n=\infty} \frac{1}{n^2}$.

Here $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$

$= \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots - \frac{1}{2^2}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$

$= \frac{\pi^2}{8} - \frac{\pi^2}{24}$ by (*g*) and (*a*) $= \frac{\pi^2}{12}$.

(*j*) Now consider the identity $\left[\sum_{n=1}^{n=\infty} \frac{1}{n^2}\right]^2 \equiv \sum_{n=1}^{n=\infty} \frac{1}{n^4} + 2 \Sigma \Sigma \frac{1}{p^2 q^2}$, where p and q are any of the numbers 1, 2, 3, ..., ∞.

Here $\left(\frac{\pi^2}{6}\right)^2 = \frac{\pi^4}{90} + 2 \Sigma \Sigma \frac{1}{q^2 q^2}$, by (*a*) and (*d*).

Then, $\Sigma \Sigma \frac{1}{p^2 q^2} = -\frac{\pi^4}{180} + \frac{\pi^4}{72} = \frac{\pi^4}{120}$. Hence the sum of the series formed by multiplying together every two of the terms of the series $\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \ldots$, is $\frac{\pi^4}{120}$.

Now consider the identity,

$$\frac{\frac{1}{2}n.(n+1)}{(2n+1)^4} = \frac{1}{8}\left[\frac{1}{(2n+1)^2} - \frac{1}{(2n+1)^4}\right].$$

Here $\sum_{n=1}^{n=\infty} \frac{\frac{1}{2}n(n+1)}{(2n+1)^4} = \frac{1}{8}\left[\sum_{n=1}^{n=\infty} \frac{1}{(2n+1)^2} - \sum_{n=1}^{n=\infty} \frac{1}{(2n+1)^4}\right]$

$= \frac{1}{8}\left[\frac{\pi^2}{8} - \frac{\pi^4}{96}\right]$ by (*g*) and (*h*) $= \frac{\pi^2}{64}\left[1 - \frac{\pi^2}{12}\right]$.

[To be continued.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

275. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Given the simultaneous equations $x^y - y^x = 0$ and $y - x = a(a+1)^{1/a}$; find a solution which is real when $a > -1$.

Solution by A. H. HOLMES, Brunswick, Maine, and L. E. NEWCOMB, Los Gatos, California.

$x^y = y^x \ldots (1)$; $y - x = a(a+1)^{1/a} \ldots (2)$. Put $y = xz$. Then in (1) $x^{xz} = x^{zx}$. $\therefore x^z = zx$. From (2), $x(x^{z-1} - 1) = a(a+1)^{1/a}$, or $x(z-1) = a(a+1)^{1/a}$. Take $(z-1) = a$. Then $x = (a+1)^{1/a}$, $z = a+1$, $y = zx = (a+1)^{(a+1)/a}$. Putting for y and x these values in (1),

$$(a+1)^{(1/a)\,(a+1)^{(a+1)/a}} = (a+1)^{[(a+1)/a](a+1)^{1/a}};$$

$$\frac{(a+1)^{a+1}}{a}\log(a+1) = \frac{a+1}{a} \times (a+1)^{1/a}\log(a+1); \text{ and}$$

$$\frac{a+1}{a} = \frac{a+1}{a}.$$

$\therefore x = (a+1)^{1/a}$ and $y = (a+1)^{(a+1)/a}$, which are real when $a > -1$.

Also solved by G. B. M. Zerr and J. Scheffer.

No solution of 276 has yet been received.

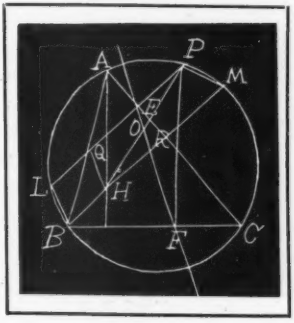
GEOMETRY.

303. Proposed by FRANCIS RUST, C. E., Allegheny, Pa.

Prove that the pedal line of any point on a triangle's circum-circle bisects the distance from this point to the triangle's ortho-center.

Solution by G. B. M. ZERR, Ph. D., Parsons, W. Va., and J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.

Let H be the ortho-center, P the given point, EF the pedal line intersecting PH in O. Produce the perpendicular BR to meet the circumference in M, and produce PE to meet the circumference in L, also take $EQ = PE$.



Quadrilateral $PEFC$ is inscriptible; $\angle PCF$ = supplement of angle PLB, both measured by $\frac{1}{2}$ arc PAB.

$\therefore \angle PEF$, the supplement of $\angle PCF = \angle PLB$. $\therefore EF$ is parallel to LB.

Now $RM = RH$, $PD = EQ$, hence the trape-

zoid $PQHM$ is isosceles. $\therefore PM=QH$, but $PM=LB$. $\therefore LB=QH$ and is parallel to it since $\angle PLB=\angle PQH=\angle PEF$. $\therefore EF$ is parallel to QH.

Now E is the midpoint of PQ, hence O is the midpoint of PH.

Also solved by the Proposer.

304. Proposed by G. W. GREENWOOD, M. A., Dunbar, Pa.

Find the tangent at the points $(a, 0)$ and $(0, a)$ to the locus $x^3+y^3=a^3$, and show that these points are points of inflexion.

I. Solution by A. H. HOLMES, Brunswick, Maine.

$x^3+y^3=a^3$. $\therefore \frac{dy}{dx}=-\frac{x^2}{(a^3-x^3)^{2/3}}$, which is 0 for $x=0$, and ∞ for $x=a$.

$\frac{d^2y}{dx^2}=-\frac{2x(a^3-x^3)^{1/3}-2x^4}{a^3-x^3}$, which is 0 for $x=0$, and ∞ for $x=a$. Take $x>a$, and $\frac{d^2y}{dx^2}$ is seen to be minus. Take $x<a$ (a little) and $\frac{d^2y}{dx^2}$ is plus.

$\therefore (a, 0)$ and $(0, a)$ are points of inflexion.

II. Solution by BENJ. F. FINKEL, Ph. D., Drury College, Springfield, Mo.

We have for the slope of the curve at any point, $\frac{dy}{dx}=-\frac{x^2}{y^2}$. $\therefore y-y_1=-\frac{x_1^2}{y_1^2}(x-x_1)$ is the equation of the tangent at any arbitrary point (x_1, y_1) of the curve. For $(0, a)$, the equation of the tangent is $y-a=0$. For $(a, 0)$, the equation of the tangent is $x-a=0$. From the equation of the tangent we have $y=y_1-\frac{x_1^2}{y_1^2}(x-x_1)$. In this equation, find y, for $x=x_1-h$ and $x=x_1+h$; also find the corresponding values of y from the equation of the curve, $y=\sqrt[3]{(a^3-x^3)}$. If the differences of these corresponding values of y change signs, the point is a point of inflection; if they do not, the point is an ordinary point of tangency. From the equation of the tangent, the values of y for the point $(0, a)$ are $y'=a$, $y''=a$, and from the curve $'y=\sqrt[3]{(a^3+h^3)}$, $''y=\sqrt[3]{(a^3-h^3)}$; $'y-y'=\sqrt[3]{(a^3+h^3)}-a>0$, $'y-y'=\sqrt[3]{(a^3-h^3)}-a<0$. Hence, $(0, a)$ is a point of inflection.

Similarly for the point $(a, 0)$, $y'=\infty$, $y''=-\infty$.

$'y=\sqrt[3]{(3a^2h-3ah^2+h^3)}$, $''y=\sqrt[3]{(-3a^2h-3ah^2-h^3)}$,

$'y-y'=\sqrt[3]{(3a^2h-3ah^2+h^3)}-\infty<0$, $''y-y''=\sqrt[3]{(-3a^2h-3ah^2-h^3)}+\infty>0$.

Hence, the point $(a, 0)$ is a point of inflection.

Also solved by G. B. M. Zerr, J. Scheffer, and the Proposer.

305. Proposed by JOHN J. QUINN, Ph. D., Scottdale, Pa.

1. Suppose two radii R and R_1 revolve uniformly in the ratio 2 : 3. Find the equation of the locus of the intersection of R with the chord drawn from the end of the diameter to the extremity of R_1. 2. If the chord be drawn to the extremity of R, what is the locus of the intersection with R_1? 3. Show how an angle can be trisected by means of this curve.

Solution by G. B. M. ZERR, Ph. D., Parsons, W. Va., and A. H. HOLMES, Brunswick, Maine.

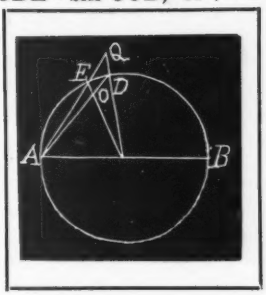
Let $CD=R$, $CE=R_1$. Then $\angle DCB=2\theta$, $\angle ECB=3\theta$, and O is the intersection of BE with R, while Q is the intersection of BD with R_1.

1. Let $CO=\rho$, $CB=a$. Then $CO : CB=\sin CBE : \sin COB$, or $\rho : a = \sin(\frac{1}{2}\pi-\frac{3}{2}\theta) : \sin(\frac{1}{2}\pi-\frac{1}{2}\theta)$. $\therefore \rho=a\cos\frac{3}{2}\theta/\cos\frac{1}{2}\theta = a(1-4\sin^2\frac{1}{2}\theta)=a(2\cos\theta-1)$, for locus of O.

2. Let $CQ=\rho'$. $\therefore \rho' : a=\sin(\frac{1}{2}\pi-\theta) : \sin(\frac{1}{2}\pi-2\theta)$, and $\rho'=a\cos\theta/\cos2\theta$, for locus of Q.

3. Through any point O, on the curve, draw BO and produce it to meet the circumference in E. Then $\angle ECO=\frac{1}{3}\angle ECB$.

Through any point Q, on the curve, draw QB intersecting the circumference in D. Then $\angle QCD=\frac{1}{3}\angle QCB$.



CALCULUS.

231. Proposed by PROF. EVA S. MAGLOTT, Ada, Ohio.

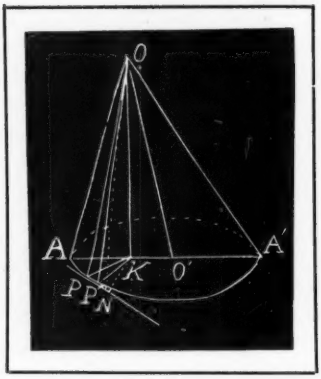
If a right circular cone stand on an ellipse, prove that the convex surface of the cone is $\frac{1}{2}\pi(OA+OA')(OA.OA')^{\frac{1}{2}}\sin\alpha$, where O is the vertex of the cone, A and A' the extremities of the major axis of the ellipse, and α is the semi-angle of the cone at the vertex, using the formula $ds=\frac{1}{2}\rho\sqrt{(\rho^2+p^2)}\,d\theta$, where p is the perpendicular from the vertex to the base of the cone, ρ the distance from the foot of the perpendicular to any point in the perimeter of the base, and θ the angle between the major axis and ρ.

Solution by BENJ. F. FINKEL, Ph. D., Drury College, Springfield, Mo.

The problem as proposed assumes that the secant of the angle between an element of the convex surface and its projection on the base is $\sqrt{(p^2+\rho^2)}/\rho$. This assumption is incorrect, and therefore the convex surface cannot be found by means of the given formula. The correct formula is $ds=\frac{1}{2}\rho^2\,d\theta\sec\gamma$, where γ is the angle between an element of the convex surface and its projection on the base of the cone. The solution is as follows:

Let $O-APA'$ be the cone, axis OO', and OK the perpendicular from the vertex to the base. Let $\alpha=\angle A'OO'=\angle AOO'$; $\beta=\angle O'OK$; $\theta=\angle AKP$; $KO=p$; $KP=\rho$; $OPP'=$an element of the convex surface; and KPP' its pro-

jection on the base of the cone.



Area of $KPP' = \frac{1}{2}\rho^2 d\theta$; $\therefore$ area $OPP' = \frac{1}{2}\rho^2 d\theta \sec\gamma$. From the triangles $O'OP$ and $O'KP$, we have $O'O^2 + OP^2 - 2O'O.OP\cos\alpha = O'K^2 + KP^2 + 2O'K.KP\cos\theta$, whence, substituting and reducing, we have

$$\rho \sin\beta\cos\theta = p\cos\beta - \sqrt{(p^2+\rho^2)}\cos\alpha \ldots (1).$$

Tangent of angle, ψ, between ρ and the tangent PP' is $\frac{\rho d\theta}{d\rho}$. Let KN be the perpendicular from K on PN. Then angle $PKN = \psi - 90°$.

$$\therefore KN = \rho\cos(\psi - 90°) = \rho\sin\psi$$

$$= \rho . \frac{\rho d\theta}{\sqrt{(d\rho^2 + \rho^2 d\theta^2)}} = \frac{\rho^2 d\theta}{\sqrt{(d\rho^2 + \rho^2 d\theta^2)}}.$$

$$\sec\gamma = \frac{ON}{KN} = \sqrt{p^2 + \left(\frac{\rho^2 d\theta}{\sqrt{(d\rho^2 + \rho^2 d\theta^2)}}\right)^2} \div \frac{\rho^2 d\theta}{\sqrt{(d\rho^2 + \rho^2 d\theta^2)}}$$

$$= \frac{1}{\rho^2}\sqrt{\rho^2\left(\frac{d\rho}{d\theta}\right)^2 + \rho^2(p^2+\rho^2)}.$$

This must be taken with a negative sign, since $\sec\gamma$ diminishes as ρ increases.

$$\therefore dS = \tfrac{1}{2}\rho^2 d\theta \sec\gamma = -\tfrac{1}{2}\sqrt{\rho^2\left(\frac{d\rho}{d\theta}\right)^2 + \rho^2(p^2+\rho^2)}\, d\theta \ldots (2).$$

From (1), $\frac{d\rho}{d\theta} = \frac{\rho\sqrt{(p^2+\rho^2)}\sin\beta\sin\theta}{\rho\cos\alpha + \sqrt{(p^2+\rho^2)}\sin\beta\cos\theta}$.

Substituting in (2) and reducing, we have $dS =$

$$-\tfrac{1}{2}p\sin\alpha\left[\frac{\rho d\rho}{\sqrt{[2p\sqrt{(p^2+\rho^2)}\cos\alpha\cos\beta - \rho^2(\cos^2\alpha - \sin^2\beta) - p^2(\cos^2\alpha + \cos^2\beta)]}}\right].$$

Let $p^2 + \rho^2 = x^2$. Substituting and reducing, we have

$$dS = -\tfrac{1}{2}p\sin\alpha\left[\frac{x dx}{\sqrt{[-\cos(\alpha-\beta)\cos(\alpha+\beta)x^2 + 2px\cos\alpha\cos\beta - p^2]}}\right].$$

The limits of ρ are AK and $A'K$. Hence, the limits of x are $\sqrt{(p^2 + AK^2)} = AO$ and $\sqrt{(p^2 + A'K^2)} = A'O$. Hence, the convex surface of the cone is

$$S=2\int_{AO}^{A'O}-\tfrac{1}{2}p\sin\alpha\left[\frac{xdx}{\sqrt{[-\cos(\alpha-\beta)\cos(\alpha+\beta)x^2+2px\cos\alpha\cos\beta-p^2]}}\right]$$

$$=-p\sin\alpha\left[\frac{\sqrt{[-\cos(\alpha-\beta)\cos(\alpha+\beta)x^2+2px\cos\alpha\cos\beta-p^2]}}{-\cos(\alpha-\beta)\cos(\alpha+\beta)}\right.$$

$$+\frac{p\cos\alpha\cos\beta}{\cos(\alpha-\beta)\cos(\alpha+\beta)}\left(\frac{1}{\sqrt{[\cos(\alpha-\beta)\cos(\alpha+\beta)]}}\right.$$

$$\left.\left.\times\sin^{-1}\left(\frac{\cos(\alpha-\beta)\cos(\alpha+\beta)x-p\cos\alpha\cos\beta}{p\sin\alpha\sin\beta}\right)\right]\right]_{AO}^{A'O}$$

$$=-\frac{p^2\sin\alpha\cos\alpha\cos\beta}{\cos(\alpha-\beta)\cos(\alpha+\beta)\sqrt{[\cos(\alpha-\beta)\cos(\alpha+\beta)]}}\left[\tfrac{1}{2}\pi-\tfrac{\pi}{2}\pi\right]$$

$$=\frac{p^2\sin\alpha\cos\alpha\cos\beta\,\pi}{\cos(\alpha-\beta)\cos(\alpha+\beta)\sqrt{[\cos(\alpha-\beta)\cos(\alpha+\beta)]}}=\frac{A'O.AO\,\pi\sin\alpha\cos\alpha\cos\beta}{\sqrt{[\cos(\alpha-\beta)\cos(\alpha+\beta)]}}.$$

Since $A'O\cos(\alpha+\beta)=AO\cos(\alpha-\beta)$, $\dfrac{A'O}{AO}=\dfrac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)}$.

$$\frac{A'O+AO}{AO}=\frac{2\cos\alpha\cos\beta}{\cos(\alpha+\beta)}.\quad\text{Also}\quad\frac{A'O+AO}{A'O}=\frac{2\cos\alpha\cos\beta}{\cos(\alpha-\beta)}.$$

$$\therefore\frac{(A'O+AO)^2}{A'O.AO}=\frac{4\cos^2\alpha\cos^2\beta}{\cos(\alpha-\beta)\cos(\alpha+\beta)},\text{ or }\frac{\cos\alpha\cos\beta}{\sqrt{[\cos(\alpha-\beta)\cos(\alpha+\beta)]}}$$

$$=\tfrac{1}{2}\frac{A'O+AO}{\sqrt{(A'O.AO)}}.$$

$$\therefore S=\tfrac{1}{2}\pi\sqrt{(A'O.AO)}\;(A'O+AO)\sin\alpha.$$

Dr. G. B. M. Zerr sent in two very simple solutions but not by projecting the convex surface on the plane of the elliptic base. Professor Scheffer sent in a solution similar to Dr. Zerr's. As the problem presents no difficulty when referred to rectangular axis with axis of the cone as one of the axes of coordinates, these solutions are omitted. The rectangular equation of the cone referred to planes $A'OA$, APA', and a plane through OK perpendicular to $A'OA$ is $y^2=\sec^2\alpha[(p-z)\sin(\alpha+\beta)+x\cos(\alpha+\beta)][(p-z)\sin(\alpha-\beta)-x\cos(\alpha-\beta)]$.

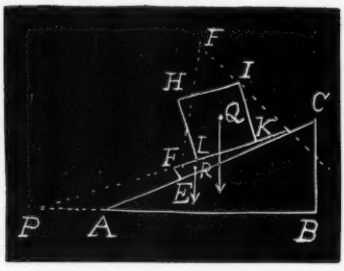
MECHANICS.

194. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

A body has a plane face resting on a rough wedge. The wedge is on a rough inclined plane, thick end down and thin edge horizontal. Find the condition that the body will slide down the wedge with constant acceleration, the wedge not slipping the while. Discuss the case in which the angle of friction for wedge and plane is greater than the angle of inclination of the plane.

Solution by G. B. M. ZERR, Ph. D., Parsons, W. Va.

Constant acceleration is a force that produces a constant increase in velocity. Gravity is such a force.



Let $\angle CBA=\beta$, the angle of inclination of the plane; $\angle FDE=\delta$, the angle of the wedge. Let ν=coefficient of friction between wedge FDE and body $HIKL$.

If $\tan DPB=\tan(\beta-\delta)>\nu$, the constant acceleration is

$$g[\sin(\beta-\delta)-\nu\cos(\beta-\delta)].$$

If $\angle FDE>\angle CAB$, then if $\tan(\delta-\beta)>\nu$, the constant acceleration is $g[\sin(\delta-\beta)-\nu\cos(\delta-\beta)]$.

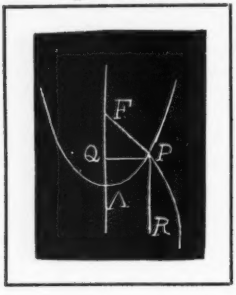
Hence, the condition for constant acceleration of the body on the wedge, the wedge not slipping, is that the coefficient of friction between body and wedge be less than the tangent of the difference of the angles of plane and wedge or wedge and plane. This covers the case for angle of friction for wedge and plane greater than angle of inclination of plane. The opposite case would form an interesting problem.

195. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Particles slide from rest at the focus of a parabola, whose axis is vertical, down radius vectors, and are then allowed to move freely. Find the locus of the foci of their subsequent paths.

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let PF be a radius vector$=r$, and $\angle PFA=\theta$. Let v be the velocity of the particle when leaving P. Draw PR parallel to AF. We find with reference to PR as the axis of abscissa, and P as origin of orthogonal co-ordinates, the equation



$$x=\frac{\cos\theta}{\sin^{\theta}}y+\frac{1}{2v^{2}\sin^{2}\theta}y^{2};$$

but $v^{2}=2gr\cos\theta$, and $r=\dfrac{a}{\cos^{2}\frac{1}{2}\theta}$. $\therefore v^{2}=\dfrac{2ag\cos\theta}{\cos^{2}\frac{1}{2}\theta}$.

$\therefore$ Equation of curve

$$y^{2}+\frac{4a\sin\theta\cos^{2}\theta}{\cos^{2}\frac{1}{2}\theta}y=\frac{4a\cos\theta\sin^{2}\theta}{\cos^{2}\frac{1}{2}\theta}x.$$

From this equation it follows that the co-ordinates of the vertex of this

parabola with reference to its axis are $\frac{a\cos^3\theta}{\cos^2\frac{1}{2}\theta}$ and $\frac{2a\sin\theta\cos^2\theta}{\cos^2\frac{1}{2}\theta}$, and its parameter $\frac{a\sin^2\theta\cos\theta}{\cos^2\frac{1}{2}\theta}$.

Denoting by x and y the current co-ordinates of this curve with reference to the axis of the given parabola, and the focus as origin. we find from the above

$$x=\frac{a\cos\theta+a\sin^2\theta\cos\theta-a\cos^3\theta}{\cos^2\frac{1}{2}\theta}=\frac{a\sin 2\theta\sin\theta}{\cos^2\frac{1}{2}\theta},$$

$$y=a\frac{\sin\theta-\sin 2\theta\cos\theta}{\cos^2\frac{1}{2}\theta}=-\frac{a\sin\theta\cos 2\theta}{\cos^2\frac{1}{2}\theta}.$$

By eliminating θ we get $(x^2+y^2-4a^2)^2(x^2+y^2)=y^2(x^2+y^2+4a^2)^2$. Introducing polar co-ordinates ρ and ϕ, we get the simpler equation

$$\rho=\pm 2a\cot(45^\circ-\tfrac{1}{2}\phi) \text{ and } \rho=\pm 2a\tan(45^\circ-\tfrac{1}{2}\phi).$$

Also solved by G. B. M. Zerr.

AVERAGE AND PROBABILITY.

160. Proposed by J. F. LAWRENCE, A. M., Stillwater, Oklahoma.

Two points are taken at random in a triangle, the line joining them dividing the triangle into two portions. Find the mean value of that portion containing the center of gravity.

Solution by HENRY HEATON, Belfield, N. D.

The triangle may be considered equilateral (see Williamson's *Integral Calculus*, p. 355). Put $\triangle$ =area of equilateral triangle whose side=a. Let P be one of the points, $PD=y$ and $BD=x$. Let EPF be the line through the two points, and put $\angle BFP=\theta$ and $\angle BEP=\phi$. Then $\phi=\frac{2}{3}\pi-\theta$, $PF=y\operatorname{cosec}\theta$ and $DF=y\cot\theta$. The area of the elemental triangle $PFF'=\frac{1}{2}y^2\operatorname{cosec}^2\theta d\theta$. We will suppose the second point to be confined to this elemental triangle. Put $BF=z$. Then $z=x+y\cot\theta$, $BE=z\sin\theta\operatorname{cosec}\phi$, and area of triangle $EBF=\frac{\triangle z^2}{a^2}\sin\theta\operatorname{cosec}\phi$.

If $\theta<\frac{1}{6}\pi$ and F is confined to the line BC the area of the portion of the triangle containing the center of gravity is $\frac{\triangle}{a^2}(a^2-z^2\sin\theta\operatorname{cosec}\theta)$. The limits of z for this are $\frac{2}{3}y\sqrt{3}\sin\phi\operatorname{cosec}\theta$ and a. Those of y are 0 and $\frac{1}{2}a\sqrt{3}\sin\theta\operatorname{cosec}\phi$. If $\theta>\frac{1}{6}\pi$ and $<\frac{1}{3}\pi$, the line EF passes through O when $z=\frac{a}{3}(1+\sin\phi\operatorname{cosec}\theta)$.

If z is less than this, y is less than $\frac{a}{6}\sqrt{3}(1+\sin\theta\ \mathrm{cosec}\phi)$.

If $z>\frac{a}{3}(1+\sin\phi\ \mathrm{cosec}\theta)$ and $y<\frac{a}{6}\sqrt{3}(1+\sin\theta\ \mathrm{cosec}\phi)$ the limits of z are $\frac{a}{3}(1+\sin\phi\ \mathrm{cosec}\theta)$ and a.

$$\Delta=\triangle\left[\int_0^{\frac{1}{6}\pi}\tfrac{1}{32}\sin\theta\,\mathrm{cosec}^3\phi-\tfrac{1}{48}\sin^2\theta\,\mathrm{cosec}^4\phi+\int_{\frac{1}{6}\pi}^{\frac{1}{3}\pi}\tfrac{1}{48}\sin^2\theta\,\mathrm{cosec}^4\phi-\frac{1}{2\cdot 3^7}\right.$$
$$\times(\sin^2\theta\,\mathrm{cosec}^4\phi+\sin^2\phi\,\mathrm{cosec}^2\theta)+\frac{1}{2^5\cdot 3^6}(\sin\theta\ \mathrm{cosec}^3\phi+\sin\phi\ \mathrm{cosec}^3\theta+\frac{41}{2^4\cdot 3^7}$$
$$\left.\times\mathrm{cosec}\theta\,\mathrm{cosec}\phi\right]d\theta\div\tfrac{1}{32}\int_0^{\frac{1}{3}\pi}\sin\theta\,\mathrm{cosec}^3\phi\,d\theta=\frac{\triangle}{3^6}(470+\tfrac{4}{3}\log 2)=.6997\,\triangle.$$

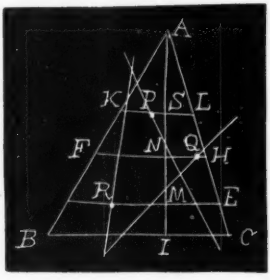
This is problem 76, p. 513, Williamson's *Integral Calculus*.

170. Proposed by LON C. WALKER, A. M., Santa Barbara, California.

Find the area of a triangle formed by drawing a line at random through each of three points taken at random within a given triangle.

Solution by G. B. M. ZERR, Ph. D., Parsons, W. Va.

Let ABC be the given triangle; P, Q, R the random points; $AI=h$, $BI=d$, $CI=e$, $d+e=a$, $AM=u$, $MR=v$, $AS=w$, $SP=z$, $AN=m$, $NQ=n$.



$y-v=r(x-u)$, the line through R... (1),
$y-z=s(x-w)$, the line through P... (2),
$y-n=t(x-m)$, the line through Q... (3),
where $r=\tan\theta$, $s=\tan\phi$, $t=\tan\psi$.

The intersection of (1) and (2) is given by

$$x_1=\frac{ru-ws+z-v}{r-s},\quad y_1=\frac{rsu-rsw+rz-sv}{r-s}.$$

The intersection of (1) and (3) is given by

$$x_2=\frac{ru-mt+n-v}{r-t},\quad y_2=\frac{rtu-mrt+rn-tv}{r-t}.$$

The intersection of (2) and (3) is given by

$$x_3=\frac{sw-mt+n-z}{s-t},\quad y_3=\frac{stw-mst+sn-tz}{s-t}.$$

Area of triangle $=\frac{1}{2}(x_2y_1-x_1y_2+x_3y_2-x_2y_3+x_1y_3-x_3y_1)$
$=\frac{1}{2}[(ur-v)(s-t)+(ws-z)(t-r)+(mt-n)(r-s)]^2/(r-s)(r-t)(s-t)$
$=A/B$.

The limits of u are 0 and h; of w, 0 and u; of m, w and u; of v, $-du/h=v_1$ and $eu/h=v_2$; of z, $-dw/h=w_1$ and $ew/h=w_2$; of n, $-dm/h=m_1$ and $em/h=m_2$. The number of ways the three points can be taken on the surface of the triangle is $\frac{1}{6}(\frac{1}{2}ah)^3=\frac{1}{48}a^3h^3$.

Hence if Δ = the required average area we get

$$\Delta = \frac{48}{a^3 h^3}\int_0^h \int_0^u \int_w^u \int_{v_1}^{v_2} \int_{w_1}^{w_2} \int_{n_1}^{n_2} \frac{A}{B}\, du\, dw\, dm\, dv\, dz\, dn$$

$$= \frac{8}{ah^8 B}\int_0^h \int_0^u \int_w^u \{(s-t)^2[3ah^2r^2+3hr(d^2-e^2)+d^3+e^3]u^3wm$$

$$+ (t-r)^2[3ah^2s^2+3hs(d^2-e^2)+d^3+e^3]uw^3m$$

$$+ (r-s)^3[3ah^2t^2+3ht(d^2-e^2)+d^3+e^3]uwm^3\}du\, dw\, dm$$

$$+ \frac{12}{a^2h^8 B}\int_0^h \int_0^u \int_w^u [(s-t)(t-r)(2ahr+d^2-e^2)(2ahs+d^2-e^2)u^2w^2m$$

$$+ (s-t)(r-s)(2ahr+d^2-e^2)(2aht+d^2-e^2)u^2wm^2$$

$$+ (t-r)(r-s)(2aht+d^2-e^2)(2ahs+d^2-e^2)uw^2m^2]du\, dw\, dm$$

$$= \frac{1}{24\,a}\left[\frac{9ah^2r^2(s-t)}{(r-s)(r-t)} + \frac{9hr(s-t)(d^2-e^2)}{(r-s)(r-t)} + \frac{3(d^3+e^3)(s-t)}{(r-s)(r-t)}\right.$$

$$+ \frac{3ah^2s^2(r-t)}{(r-s)(s-t)} + \frac{3hs(r-t)(d^2-e^2)}{(r-s)(s-t)} + \frac{(d^3+e^3)(r-t)}{(r-s)(s-t)}$$

$$\left. + \frac{6ah^2t^2(r-s)}{(s-t)(r-t)} + \frac{6ht(r-s)(d^2-e^2)}{(s-t)(r-t)} + \frac{2(d^3+e^3)(r-s)}{(s-t)(r-t)}\right]$$

$$+ \frac{1}{60a^2}\left[\frac{24\,a^2h^2rs}{s-r} + \frac{36\,a^2h^2rt}{r-t} + \frac{20\,a^2h^2st}{t-s} + \frac{12ah(d^2-e^2)(r+s)}{s-r}\right.$$

$$+ \frac{18ah(d^2-e^2)(r+t)}{r-t} + \frac{10\,ah(d^2-e^2)(s+t)}{t-s} + \frac{6\,(d^2-e^2)}{s-r} + \frac{9(d^2-e^2)^2}{r-t}$$

$$\left. + \frac{5(d^2-e^2)^2}{t-s}\right] = \frac{1}{24\,a}C + \frac{1}{60\,a^2}D.$$

The limits of θ, ϕ, and ψ are for each 0 and $\frac{1}{2}\pi$ and doubled.

Now $\dfrac{r^2(s-t)}{(r-s)(r-t)} = \dfrac{r^2}{r-s} - \dfrac{r^2}{r-t}$

$$\therefore \int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{r^2(s-t)}{(r-s)\ (r-t)}d\theta\,d\phi\,d\psi=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{s^2(r-t)}{(r-s)\ (s-t)}d\theta d\phi d\psi$$

$$=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{t^2(r-s)}{(s-t)\ (r-t)}d\theta\,d\phi\,d\psi=0.$$

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{r(s-t)}{(r-s)\ (r-t)}d\theta\,d\phi\,d\psi=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{s(r-t)}{(r-s)\ (s-t)}d\theta\,d\phi\,d\psi$$

$$=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{t(r-s)}{(s-t)\ (r-t)}d\theta\,d\phi\,d\psi=0.$$

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{(s-t)}{(r-s)\ (r-t)}d\theta\,d\phi\,d\psi=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{(r-t)}{(r-s)\ (s-t)}d\theta\,d\phi\,d\psi$$

$$=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{(r-s)}{(s-t)\ (r-t)}d\theta\,d\phi\,d\psi=0. \quad \therefore C=0.$$

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{rs}{r-s}d\theta\,d\phi\,d\psi=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{rt}{r-t}d\theta\,d\phi\,d\psi$$

$$=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{st}{t-s}d\theta\,d\phi\,d\psi=\tfrac{1}{8}\pi^2.$$

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{r+s}{r-s}d\theta\,d\phi\,d\psi=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{r+t}{r-t}d\theta\,d\phi\,d\psi$$

$$=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{s+t}{t-s}d\theta\,d\phi\,d\psi=0.$$

$$\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{d\theta\,d\phi\,d\psi}{s-r}=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{d\theta\,d\phi\,d\psi}{r-t}=\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi}\frac{d\theta\,d\phi\,d\psi}{t-s}=-\tfrac{1}{8}\pi^2.$$

$$\therefore D=\frac{10\pi^2a^2h^2-\frac{5}{2}\pi^2(d^2-e^2)^2}{\frac{1}{8}\pi^3}.$$

$$\therefore \Delta=\frac{4a^2h^2-(d^2-e^2)^2}{3\pi a^2}=\frac{[a^2(2c^2+2b^2-a^2)\ -\ 2(c^2-b^2)^2]}{3\pi a^2}.$$

If $a=b=c$, $\Delta=a^2/\pi$.

PROBLEMS FOR SOLUTION.

ALGEBRA.

180. Proposed by R. D. CARMICHAEL, Anniston, Ala.

Find values of x, y, z, and u satisfying the equations

$$x+y+z+u=10\ldots[1],$$
$$x^2+y^2+z^2+u^2=30\ldots[2],$$
$$x^3+y^3+z^3+u^3=100\ldots[3],$$
$$x^4+y^4+z^4+u^4=354\ldots[4].$$

181. Proposed by A. H. HOLMES, Brunswick, Maine.

Sum the series, $1+2^m+3^m+4^m+\ldots+n^m$.

182. Proposed by O. L. CALLICOTT, Gettysburg, S. Dak.

Find the value of $\sqrt{2}\ \sqrt[3]{2}\ \sqrt[4]{2}\ \sqrt[5]{2}\ldots\sqrt[1000]{2}$.

GEOMETRY.

313. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Prove that an algebraic curve of odd degree which is symmetrical with respect to a center has the center on the curve.

314. Proposed by F. ANDEREGG, A. M., Professor of Mathematics, Oberlin College, Oberlin, Ohio.

Find the area of the triangle bounded by the lines $l\alpha+m\beta+n\gamma=0$; $l'\alpha+m'\beta+n'\gamma=0$; $l''\alpha+m''\beta+n''\gamma=0$, where α stands for $x\cos\alpha+y\sin\alpha-p$, etc. [See Salmon's *Conic Sections*, 6th ed.]

315. Proposed by ROBERT E. MORITZ, Ph. D., University of Washington.

Given the area of the segment of a circle of given radius to find the length of the chord.

CALCULUS.

237. *Prize Problem. Proposed by S. A. COREY, Hiteman, Iowa.

Find an expression for the remainder after n terms in the following development† of $f(a+x)$:

$$f(a+x)=f(a)+\frac{x}{m.2}\Big\{f'(a+x)+f'(a)+2\Big[f'\Big(a+\frac{x}{m}\Big)+f'\Big(a+\frac{2x}{m}\Big)+f'\Big(a+\frac{3x}{m}\Big)+\ldots+f'\Big(a+\frac{m-1}{m}x\Big)\Big]\Big\}-\frac{B_1x^2}{m^2.2!}\Big[f''(a+x)-f''(a)\Big]+\frac{B_2x^4}{m^4.4!}\Big[f^{\mathrm{iv}}(a+x)-f^{\mathrm{iv}}(a)\Big]-\ldots+(-1)^n\frac{B_nx^{2n}}{m^{2n}.(2n)!}\Big[f^{(2n)}(a+x)-f^{(2n)}(a)\Big]+\ldots,$$

B_1, B_2, ... being Bernoulli's numbers.

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*In order to emphasize the importance of finding such an expression for the remainder after n terms as will hold good for all integral values of m and approach 0 as m approaches ∞, and at the same time enable computers to determine absolutely in what cases the development holds, the proposer offers a prize of $10 for the best solution. ED. F.

†See *Annals of Mathematics*, Second Series, Vol. 5, No. 4, July, 1904.

238. Proposed by PROF. R. D. CARMICHAEL, Anniston, Ala.

Solve the differential equations

$$(a)\ (x+y^2)\,dx+(x^2+y)\,dy=0,$$
$$(b)\ (x+xy+y^2)\,dx-(x^2+xy-y)\,dy=0.$$

MECHANICS.

200. Proposed by C. N. SCHMALL, 89 Columbia Street, New York City.

An elastic string whose weight is W is laid over the top of an inclined plane so as to remain at rest. Determine how much the string will be elongated, knowing, M=modulus of elasticity, L=normal length of string, and ϕ=inclination of the plane.

201. Proposed by G. B. M. ZERR, Ph. D., Parsons, W. Va.

ABC is an inclined plane, perfectly rough, length $AC=l$. The time for a sphere to roll down when AB is base is to the time for a cylinder to roll down when BC is base as m is to n. Find AB and BC.

AVERAGE AND PROBABILITY.

187. Proposed by HENRY HEATON, Belfield, N. D.

Through every point of a given square straight lines are drawn in every possible direction, terminating in the sides of the square. What is the average length of such lines?

188. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

Find the average length of a hole at random through a given (a) sphere, (b) cube.

MISCELLANEOUS.

169. Proposed by E. D. ROE, Ph. D., Syracuse University, Syracuse, N. Y.

Find the value for all finite values of k of

$$\operatorname*{Lim.}_{x=\infty}\left[x^k \log\left(\frac{e^x+1}{e^x-1}\right)\right].$$

170. Proposed by J. W. NICHOLSON, A. M., LL. D., Baton Rouge, La.

If n and m are any two real numbers whatever, n being less than m, find a rational r such that $\sqrt{n}<r<\sqrt{m}$.

NOTES AND NEWS.

George William Greenwood has been appointed professor of mathematics at Roanoke College, Salem, Va. S.

The University of Washington, Seattle, announces the establishment of five teaching fellowships in mathematics, each yielding annual stipends from $400 to $500. Fellowships are open to graduate students only.

The twenty-first regular meeting of the Chicago Section of the American Mathematical Society will be held at The University of Chicago, Saturday, March 30, 1907. Abstracts of papers to be presented at this meeting should be in the hands of the Secretary not later than March 15. S.

"Introduction to Infinitesimal Analysis" is a new book from the press of John Wiley and Sons, by Oswald Veblen and N. J. Lennes. It is a valuable contribution to the literature available in English for the student of Advanced Calculus. It will be reviewed in a later issue of the MONTHLY. S.

It will be of interest to present readers of the MONTHLY to know that Professor Leonard E. Dickson, the retiring editor, contributed the first mathematical paper to these columns, Vol. I, No. 1, pp. 7-11, January, 1894. Since that time his interest has been continuous and his contributions numerous. S.

A preliminary meeting, to organize a Rochester, N. Y., Section of the Association of Teachers of Mathematics in the Middle States and Maryland, was held on February 23. This is in line with the rapidly increasing interest throughout the country in the question of bettering the teaching of mathematics, and it is most gratifying to see both colleges and secondary schools co-operating in this good cause. S.

At the last meeting of the Mathematics Section of the Central Association of Science and Mathematics Teachers, an important report on the Teaching of Geometry was presented by a committee of five who had been at work on the subject during the preceding year. This report is now in press, and may be had by enclosing a two cent stamp to the Secretary, Miss Mabel Sykes, 438 57th Street, Chicago, Ill. S.

BOOKS AND PERIODICALS.

The Teaching of Mathematics in the Elementary and the Secondary Schools. By J. W. A. Young, Ph. D., Assistant Professor of the Pedagogy of Mathematics in The University of Chicago. 8vo Cloth, xviii+350 pages. Price, $1.50.

This volume constitutes a very valuable addition to the Pedagogical Literature of Mathematics. The book is just from the press, and we have had time to read only the first three chapters and make a cursory examination of the remainder of the work. We have found the book so interesting and suggestive that we hasten to call the attention of our readers to it. The author tells of conditions as they are and states facts that are easily verified by investigation and experience. Too many works on general pedagogy describe conditions that are purely Utopian and that can never be realized so long as there are human beings to be educated. Thus, the author says, p. 11, "Some pupils no doubt regard the whole process of education (or any particular subject) as a set of tasks intended in some undefined way for the gratification of others, and consider that their own best interests lie in evading as far as possible the execution of these tasks." What teachers are there that have not observed this fact? Yet there are many so called teachers that refuse to acknowledge it. What teacher has not observed that most of the schools and colleges have just two classes of students? First, those who try to get as much out of their work as they can, and second, those who try to get out of as much of their work as they can. It is useless to try to write text books suitable for the latter class. Books composed of fly leaves would be the most satisfactory to them. The author also calls attention to the very patent fact that bad results in the teaching of mathematics are often due to the lack of special preparation on the part of the teacher. Thus, he says, p. 3, "It is even not unknown that classes in mathematics have been confided to teachers of other subjects, having neither special preparation for teaching mathematics nor experience in it, for no other reason than that they had a vacant period." It is a fact that teachers are too often selected without the slightest reference to their fitness or preparation for their work. This is true not only in mathematics but in other subjects as well.

The work contains the following fifteen chapters: The Study of the Pedagogy of Mathematics; The Purpose and Value of the Study of Mathematics in Primary and Secondary Schools; Methods and Modes; The Heuristic Method; The Individual Mode; The Perry Movement,—The Laboratory Method; Miscellaneous Points of Method and Mode; Preparation of Teachers,—Mathematical Clubs; The Material Equipment; The Curriculum in Mathematics; Definitions and Axioms; The Teaching of Arithmetic; The Teaching of Geometry; The Teaching of Algebra; Limits.

The true teacher of elementary mathematics will read this book with interest and profit. B. F. F.

Self Propelled Vehicles. A Practical Treatise on the Theory, Construction, Operation, Care, and Management of All Forms of Automobiles. By James E. Homans, A. M. 8vo Cloth, 598 pages. New York: Theo. Audel & Co.

This is the sixth edition of a work of great value to all interested in the automobile. It also contains much information of great value to the practical mechanic. B. F. F.

The Foundations of Higher Arithmetic. By B. F. Fisk, M. S., Instructor in Senior Arithmetic in the Austin High School. 8vo Cloth, vi+203 pages. New York: Silver, Burdett & Co.

This book is a step in the right direction of the teaching of the subject. The solutions are arranged in order, and are clear and concise. B. F. F.

The Elements of Geometry. By Walter N. Bush, Principal of the Polytechnic High School, San Francisco, and John B. Clarke, Department of Mathematics, Polytechnic High School, San Francisco. 8vo Cloth, x+355 pages. New York: Silver, Burdett & Co.

The treatment of Geometry in this book is along the line usually followed in American works. A large collection of originals are scattered throughout the book. B. F. F.

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SUMMATION OF CERTAIN INFINITE SERIES.

By W. J. GREENSTREET, M. A., F. R. A. S., Editor of The Mathematical Gazette, Stroud, England.

[Concluded from February Number.]

(*l*) To get a series such has $\Sigma\Sigma\frac{1}{(2m+1)^2(2n+1)^2}$, where m and n may have any integral values from 1 to ∞, we proceed as follows:

$$\left[\sum_{n=1}^{n=\infty}\frac{1}{(2n-1)^2}\right]^2=\sum_{n=1}^{n=\infty}\frac{1}{(2n-1)^4}+2\Sigma\Sigma\frac{1}{(2p+1)^2(2q+1)^2}.$$

$$\therefore \Sigma\Sigma\frac{1}{(2p+1)^2(2q+1)^2} \text{ (where } 2p+1 \text{ and } 2q+1 \text{ are any odd numbers from 1 to } \infty) = \tfrac{1}{2}\left[\left(\frac{\pi^2}{8}\right)^2-\frac{\pi^4}{96}\right] \text{ by } (g) \text{ and } (h) = \frac{\pi^4}{384}.$$

(*m*) Consider the identity $\frac{1}{n^2(n+1)^2}\equiv\frac{1}{n^2}+\frac{1}{(n+1)^2}-2\left(\frac{1}{n}-\frac{1}{n+1}\right)$.

$$\text{Here } \sum_{n=1}^{n=\infty}\frac{1}{n^2(n+1)^2}=\sum_{n=1}^{n=\infty}\frac{1}{n^2}+\left[\sum_{n=1}^{n=\infty}\frac{1}{n^2}-1\right]-2(1-\tfrac{1}{2}+\tfrac{1}{2}-\tfrac{1}{3}+\tfrac{1}{3}-\tfrac{1}{4}+\ldots)$$

$$=\frac{\pi^2}{6}+\frac{\pi^2}{6}-1-2 \text{ by } (a) =\frac{\pi^2}{3}-3.$$

(*n*) Similarly, $\frac{1}{n^3(n+1)^3}\equiv\frac{1}{n^3}-\frac{1}{(n+1)^3}-\frac{3}{n^2(n+1)^2}$ gives

$$\sum_{n=1}^{n=\infty}\frac{1}{n^3(n+1)^3}=\left(\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\ldots\right)-\left(\frac{1}{2^3}+\frac{1}{3^3}+\ldots\right)-3\sum_{n=1}^{n=\infty}\frac{1}{n^2(n+1)^2}$$

$$=1-3\left(\frac{\pi^2}{3}-3\right)=10-\pi^2, \text{ by } (m).$$